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ON THE DISTRIBUTION OF THE ZEROS OF A POLYNOMIAL

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ABSTRACT

Let P(z) be a polynomial of degree n with real or complex coefficients. In this paper, we shall obtain several generalizations and extensions of a well-known result of Enestrom and Kakeya about the location of the zeros of a polynomial.

KEYWORDS Zeros, Coefficients, Enestron-Kakeya Theorem. Mathematics Subject Classification(2000) 30C10,30C15.

INTRODUCTION

Introduction and Statement of Results

The following well-known result in the theory of distribution of the zeros of polynomials is due to Enestrom and Kakeya (for reference see [1])

Theorem A (Enestrom-Kakeya). If

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0,$$

Is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0 \qquad (1)$$

Then all the zeros of P(z) lie in $|z| \leq 1.$

This result was proved by Enestrom[5], independently by Kakeya[9] and Hurwitz[7].we now apply this result to P(tz) to obtain the following more general result:

Theorem B. If

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0,$$

Is a polynomial of degree n such that

$$a_nt^n \geq a_{n-1}t^{n-1} \geq ... \geq a_1t \geq a_0 > 0 \qquad (2)$$

Then all the zeros of P(z) lie in $|z| \leq t.$

This theorem has been extended and sharpened in various ways(see Krishnala[10],Cargo and Shisha [3] Joyal,Labelle and Rehman [8], Govil and Rahman [6] etc.).

As a compact generalization of Theorems A and B Aziz and Mohammad [1] have used Schwarz lemma and proved:

Theorem C. If

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0,$$

Is a polynomial of degree n with real and positive coefficients .If $t_1 \geq t_2 \geq 0$ Can be found such that

$$a_rt_1t_2+a_{r-1}(t_1-t_2)+a_{r-2} \geq 0, \qquad r=1,2,...,n+1. \qquad (3)$$

Where $(a_{-1}=a_{n+1}=0)$ then all the zeros of P(z) lie in $|z| \leq t_1.$

For $t_2=0,$ this reduces to Theorem B and for $t_1=1, t_2=0,$ this reduces to Enestrom- Kakeya Theorem.

Taking monotonicity of the coefficients of a polynomial Joyal, Labelle and Rahman [8] proved:

Theorem D. Let

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$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,$$

Is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} \qquad (4).$$

As a generalization of Theorem D . Dewan and Bidkham [4] have obtained the following result:

Theorem E. Let

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,$$

be a polynomial of degree n, such that t>0 and $0 \leq k \leq n,$

$$a_n t^n \leq a_{n-1}t^{n-1} \leq ... \leq t^k a_k \geq t^{k-1}a_{k-1} \geq ... \geq a_1 t \geq a_0 \qquad (5)$$

then all the zeros of P(z) lie in the circle $|z| \leq \frac{t}{|a_n|}\left\{(\frac{2a_k}{t^{n-k}} - a_n) + \frac{1}{t^n}(|a_0| - a_0)\right\}.$ (6)

The main aim of this paper is to establish a compact generalization of Theorems C and E and an extension of Theorem E. We first present:

Theorem 1.1. Let

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,$$

be a polynomial of degree n with real and positive coefficients .If $t_1 \neq 0$ *and* t_2 be found such that $t_1 \geq t_2 \geq 0$,

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) + a_{r-2} \geq 0, \qquad r = 1,2,...,n+1. \qquad (7)$$

and

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) + a_{r-2} \leq 0, \qquad r = k+2,....,n+1.$$

Where $(a_{-1} = a_{n+1} = 0)$ then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\left\{t_1(\frac{2a_k}{t_1^{n-k}} - a_n) + \frac{(t_1 + t_2)}{t_1^n}(|a_0| - a_0) + t_2(\frac{2a_{k+1}}{t_1^{n+1}} - a_n + |a_n|)\right\}. \qquad (8)$$

Remark 1. Let

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,$$

be a polynomial of degree n with real and positive coefficients .Taking k=n in Theorem 1.1 and noting that $(|a_0| - a_0)$ =0, $(|a_n| - a_n) = 0$, . So that

$$\frac{1}{|a_n|}(\frac{2a_k}{t_1^{n-k}} - a_n) = \frac{2a_n - a_n}{|a_n|} = \frac{a_n}{|a_n|} = 1.$$

It follows that all the zeros of P(z lie in $|z| \leq t_1$. which is precisely Theorem C.

Remark 2. If we take $t_2 = 0$ and .If $t_1 = t$ in Theorem 1.1 we get Theorem E.
Next, we shall present the following extension of Theorem E.

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Theorem 1.2. . Let

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z \ + a_0,$$

be a polynomial of degree n, such that for some t>0 and $0 \le k \le n,$

$$a_{n-1}t^{n-1} \le a_{n-2}t^{n-2} \le ... \le t^{k-1}a_k \ge t^k a_{k+1} \ge ... \ge a_2 t \ge a_1 \qquad (9)$$

and if a_0 is any real or complex number,then all the zeros of P(z) lie in the circle

$$|z| \le \frac{t}{|a_n|}\left\{(\frac{2a_k}{t^{n-k}} - a_n) + \frac{1}{t^n}(|a_0| - ta_1 + |ta_1 - a_0|)\right\}. \qquad (10)$$

Remark 3. If a_0 is real such that $a_1 \ge \frac{a_0}{t}$,then we immediately get theorem E.

If we take t=1 and k=n in Theorem 1.2, we get the following result:

Corollary 1. Let

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z \ + a_0,$$

be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ... \ge a_1 \ge a_0,$$

and if a_0 is any real or complex number, then all the zeros of P(z) lie in

$$|z| \le \frac{a_n - a_0 + |a_0| + |a_1 - a_0|}{|a_n|} \ .$$

If $a_1 \ge a_0$, then corollary 1 reduces to Theorem D.

For the proofs of these theorems we need the following lemma ,which is due to Aziz and Mohammad[2].

Lemma. Let

$$P(z)=a_n z^n + a_p z^p + ... + a_1 z \ + a_0 \qquad 0 \le p \le n-1,$$

be a polynomial of degree n with complex coefficients, then for every real number r , all the zeros of P(z) lie in the circle

$$|z| \le Max\left\{r, \sum_{j=0}^{p} \frac{a_j}{a_n}\frac{1}{r^{n-j-1}}\right\}.$$

Proofs of Theorems

Proof of Theorem 1.1. Consider the polynomial

$$\begin{aligned} G(z) &= (t_1 - z)(t_2 - z)P(z) \\ &= (t_1 t_2 + (t_1 - t_2)z - z^2)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z \ + a_0) \\ &= -a_n z^{n+2} + \{(t_1 - t_2)a_{n-1} - a_{n-2}\}z^{n+1} + ... + \{a_2 t_1 t_2 + (t_1 - t_2)a_1 - a_0\}z^2 \\ &\quad + \{a_1 t_1 t_2 + (t_1 - t_2)a_0\}z + a_0 t_1 t_2 \\ &= -a_n z^{n+2} + \sum_{j=0}^{n+1}\{a_j t_1 t_2 + (t_1 - t_2)a_{j-1} - a_{j-2}\}z^j \ , (a_{-1} = a_{n+1} = 0) \end{aligned}$$

Since G(z) is a polynomial of degree n+2.Applying the above lemma, it follows that all the zeros of G(z) lie in the circle

$$|z| \le Max\left\{ t_1, \sum_{j=0}^{n+1} \frac{a_j t_1 t_2 + (t_1 - t_2) a_{j-1} - a_{j-2}}{|a_n|} \frac{1}{t_1^{\,n-j-1}} \right\}.$$

Now,

$$\left| \sum_{j=0}^{n+1} \frac{a_j t_1 t_2 + (t_1 - t_2) a_{j-1} - a_{j-2}}{|a_n|} \frac{1}{t_1^{\,n-j-1}} \right|$$

$$= \frac{1}{|a_n|} \left| \sum_{j=0}^{n+1} \left[\frac{a_j t_2}{t_1^{\,n-j}} + \frac{a_{j-1}}{t_1^{\,n-j}} - \frac{a_{j-1} t_2}{t_1^{\,n-j+1}} - \frac{a_{j-2}}{t_1^{\,n-j+1}} \right] \right|$$

$$= \frac{1}{|a_n|} \left| \sum_{j=0}^{n+1} t_2 \left[\frac{a_j}{t_1^{\,n-j}} - \frac{a_{j-1}}{t_1^{\,n-j+1}} \right] + \sum_{j=0}^{n+1} \left[\frac{a_{j-1}}{t_1^{\,n-j}} - \frac{a_{j-2}}{t_1^{\,n-j+1}} \right] \right|$$

$$= t_1$$

Since

$$t_1 = \left| \sum_{j=0}^{n+1} \frac{a_j t_1 t_2 + (t_1 - t_2) a_{j-1} - a_{j-2}}{|a_n|} \frac{1}{t_1^{\,n-j-1}} \right|$$

$$\le \sum_{j=0}^{n+1} \left| \frac{a_j t_1 t_2 + (t_1 - t_2) a_{j-1} - a_{j-2}}{|a_n| t_1^{\,n-j-1}} \right|$$

But

$$\left| \sum_{j=0}^{n+1} \frac{a_j t_1 t_2 + (t_1 - t_2) a_{j-1} - a_{j-2}}{|a_n|} \frac{1}{t_1^{\,n-j-1}} \right|$$

$$\le \frac{1}{|a_n|} \sum_{j=0}^{n+1} t_2 \left| \frac{t_1 a_j}{t_1^{\,n-j+1}} - \frac{a_{j-1}}{t_1^{\,n-j+1}} \right| + \sum_{j=0}^{n+1} \frac{1}{|a_n|} \left| \frac{t_1 a_{j-1}}{t_1^{\,n-j+1}} - \frac{a_{j-2}}{t_1^{\,n-j+1}} \right|$$

$$= \sum_{j=0}^{k+1} \frac{t_2}{|a_n|} \left| \frac{t_1 a_j}{t_1^{\,n-j+1}} - \frac{a_{j-1}}{t_1^{\,n-j+1}} \right| + \sum_{j=k+2}^{n+1} \frac{t_2}{|a_n|} \left| \frac{t_1 a_j}{t_1^{\,n-j+1}} - \frac{a_{j-1}}{t_1^{\,n-j+1}} \right|$$

$$+ \sum_{j=0}^{k+1} \frac{1}{|a_n|} \left| \frac{t_1 a_{j-1}}{t_1^{\,n-j+1}} - \frac{a_{j-2}}{t_1^{\,n-j+1}} \right| + \sum_{j=k+2}^{n+1} \frac{1}{|a_n|} \left| \frac{t_1 a_{j-1}}{t_1^{\,n-j+1}} - \frac{a_{j-2}}{t_1^{\,n-j+1}} \right|$$

$$=$$

$$\frac{t_2}{|a_n|} \left\{ \frac{|a_0|}{t_1^{\,n}} - \frac{a_1}{t_1^{\,n-1}} - \frac{a_0}{t_1^{\,n}} + \frac{a_2}{t_1^{\,n-2}} - \frac{a_1}{t_1^{\,n-1}} - \ldots + \frac{a_k}{t_1^{\,n-k}} - \frac{a_{k-1}}{t_1^{\,n-k+1}} + \frac{a_{k+1}}{t_1^{\,n-k+1}} - \frac{a_{k+2}}{t_1^{\,n-k+2}} + \ldots + \frac{a_{n-2}}{t_1^{\,2}} - \frac{a_{n-1}}{t_1} - \frac{a_{n-2}}{t_1} - a_n + |a_n| \right\} +$$

$$\frac{1}{|a_n|} \left\{ \frac{|a_0|}{t_1^{\,n-1}} + \frac{a_1}{t_1^{\,n2}} - \frac{a_0}{t_1^{\,n-1}} + \ldots + \frac{a_k}{t_1^{\,n-k-k}} - \frac{a_{k-2}}{t_1^{\,n-k+1}} + \frac{a_{k+1}}{t_1^{\,n-k}} - \frac{a_{k+2}}{t_1^{\,n-k+1}} + \ldots + \frac{a_{n-2}}{t_1} - a_{n-1} + a_{n-1} - a_n t \right\}$$

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$$= \left\{ t_1\left(\frac{2a_k}{t_1^{n-k}} - a_n\right) + \frac{(t_1+t_2)}{t_1^{\,n}}(|a_0| - a_0) + t_2\left(\frac{2a_{k+1}}{t_1^{n+1}} - a_n + |a_n|\right) \right\}$$

Since all the zeros of P(z) are also the zeros of G(z) ,it follows that all the zeros P(z) lie in the circle defined by (8) and this completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Consider the polynomial

$$F(z) = (t-z)P(z)$$
$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + \ldots + (ta_1 - a_0)z + a_0 t$$
$$= -a_n z^{n+1} + \sum_{j=0}^{n}(ta_j - a_{j-1})z^j \qquad (a_{-1} = 0)$$

Since F(z) is a polynomial of degree n+1, using the above lemma to F(z) with p=n and r=t, it follows that all the zeros of F(z) lie in the circle

$$|z| \le Max\left\{ t, \sum_{j=0}^{n}\frac{|ta_j - a_{j-1}|}{|a_n|}\frac{1}{t^{n-j}} \right\}$$
$$= \sum_{j=0}^{n}\frac{|ta_j - a_{j-1}|}{|a_n|}\frac{1}{t^{n-j}}$$

Since

$$t = \left| \sum_{j=0}^{n}\frac{(ta_j - a_{j-1})}{|a_n|}\frac{1}{t^{n-j}} \right| \le \sum_{j=0}^{n}\frac{|ta_j - a_{j-1}|}{|a_n|}\frac{1}{t^{n-j}}$$
$$= \sum_{j=0}^{k}\frac{|ta_j - a_{j-1}|}{|a_n|}\frac{1}{t^{n-j}} + \sum_{j=k+1}^{n}\frac{|ta_j - a_{j-1}|}{|a_n|}\frac{1}{t^{n-j}}$$
$$= \frac{t}{|a_n|}\left\{ \left(\frac{2a_k}{t^{n-k}} - a_n\right) + \frac{1}{t^n}(|a_0| - ta_1 + |ta_1 - a_0|) \right\} \text{ (by hypothesis)}$$

Hence all the zeros of F(z) lie in

$$|z| \le \frac{t}{|a_n|}\left\{ \left(\frac{2a_k}{t^{n-k}} - a_n\right) + \frac{1}{t^n}(|a_0| - ta_1 + |ta_1 - a_0|) \right\} .$$

As all the zeros of P(z) are also the zeros of F(z), the result follows and hence Theorem 1.2 is proved completely.

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